Exercise 1: Three-Level System

We consider the interaction of a three-level system with two light fields using the density-matrix formalism. The important quantities are noted in the following diagram:

The Hamiltonian of the system and the density matrix are given by:

\[
\hat{H} = -\frac{\hbar}{2} \begin{pmatrix}
0 & 0 & \Omega_p \\
0 & 2(\delta_1 - \delta_2) & \Omega_c \\
\Omega_p & \Omega_c & 2\delta_1
\end{pmatrix}
\]

and

\[
\dot{\rho} = \begin{pmatrix}
\rho_{1,1} & \rho_{1,2} & \rho_{1,3} \\
\rho_{2,1} & \rho_{2,2} & \rho_{2,3} \\
\rho_{3,1} & \rho_{3,2} & \rho_{3,3}
\end{pmatrix}.
\]

The time-evolution follows the von Neumann equation

\[
\frac{\partial \rho}{\partial t} = -\frac{i}{\hbar} \left[ \hat{H}, \rho \right] + \begin{pmatrix}
\Gamma_1 \rho_{3,3} & 0 & -\frac{1}{2} \rho_{1,3} (\Gamma_1 + \Gamma_2) \\
0 & \Gamma_2 \rho_{3,3} & -\frac{1}{2} \rho_{2,3} (\Gamma_1 + \Gamma_2) \\
-\frac{1}{2} \rho_{3,1} (\Gamma_1 + \Gamma_2) & -\frac{1}{2} \rho_{3,2} (\Gamma_1 + \Gamma_2) & -\rho_{3,3} (\Gamma_1 + \Gamma_2)
\end{pmatrix}
\]  \hspace{1cm} (1)

In this exercise sheet we will find the stationary solutions to this system of equations and use the solutions to understand the effects described in:
Read both papers for discussion in the group and work out the following exercises:

a) Solve the system of equations (1) for the stationary case \( \frac{\partial \hat{\rho}}{\partial t} = 0 \) and for a resonant coupling laser \( (\delta_2 = 0) \) leaving the frequency of the probe laser \( \delta_1 \) as a variable and the Rabi frequencies \( (\Omega_p \text{ and } \Omega_c) \) as parameters. You can use a mathematical program for this.

b) Derive the expression for the group velocity \( v_g \) (the exact equality in equation (1) in the 1999 paper).

Note: The group velocity is defined by \( v_g = \frac{\partial \omega}{\partial k} \). To obtain the expression consider \( \frac{\partial k}{\partial \omega} \).

c) The complex index of refraction \( n = n + ik \) which affects the probe laser can be obtained by \( n = \sqrt{1 + \chi} \), where \( \chi \) is the electric susceptibility given by

\[
\chi = \frac{N |\mu_{13}|}{V \epsilon_0 E_0} \rho_{31}.
\]

Here \( N/V \) is the atom number density, \( \mu_{13} \) is the transition matrix dipole element between state \( |1\rangle \) and \( |3\rangle \), \( \epsilon_0 \) is the vacuum permittivity and \( E_0 \) is the amplitude of the applied electric field.

Using your solution from a), determine the real and imaginary parts of the refractive index and plot them vs \( \delta_1 \). (See Figure 2a in the paper).

Note: Useful relations: \( \Omega_p = \mu_{13}E_0/\hbar \) and \( I/I_{sat} = 2 (\Omega/\Gamma)^2 \).

d) Finally, calculate the propagation velocity of the probe light at resonance \( (\delta_1 = 0) \).

e) Now consider an additional dephasing process between the states \( |1\rangle \) and \( |2\rangle \). The time evolution for this system is

\[
\frac{\partial \hat{\rho}}{\partial t} = -\frac{i}{\hbar} \left[ \hat{H}, \hat{\rho} \right] + \begin{pmatrix}
\Gamma_1 \rho_{3,3} & -\frac{1}{2} \rho_{1,2} \gamma_{12} & -\frac{1}{2} \rho_{1,3} (\Gamma_1 + \Gamma_2) \\
-\frac{1}{2} \rho_{2,1} \gamma_{12} & \Gamma_2 \rho_{3,3} & -\frac{1}{2} \rho_{2,3} (\Gamma_1 + \Gamma_2) \\
-\frac{1}{2} \rho_{3,1} (\Gamma_1 + \Gamma_2) & -\frac{1}{2} \rho_{3,2} (\Gamma_1 + \Gamma_2) & -\rho_{3,3} (\Gamma_1 + \Gamma_2)
\end{pmatrix}
\]

finding the stationary solutions for this case, plot the resulting real and imaginary parts of the index of refraction of the medium.